OPTIMIZATION OF THE ADM BY ADAPTIVE WEIGHTING FOR THEGAUSSIAN QUADRATURE INTEGRATION

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ABSTRACT

In order to determine the flow rate through a pipe with the acoustic transit time method, integration of the measured mean projected velocities on the acoustic paths is needed. Adaptive integration methods may increase considerably accuracy of the integrated flow rate. Since velocity can be mapped onto discharge with an integral operator, the discrete output of an ADM (Acoustic Discharge Measurement) must be integrated numerically. One common method to do this for pipe flows is called OWICS method. OWICS uses a fixed parameter to describe the flow field, which might not be best for any flow situation. Thus, this paper introduces a dynamic OWICS method, which adaptively determines the flow parameter needed for integration. The presented examples show that discharge calculation can be improved with an adaptive scheme by one fifth up to one half. However, the magnitude of improvement depends strongly on how the velocity profile is disturbed. The adaptive scheme can handle disturbances, which are somewhat symmetric about the center axis, more reliable than ones that are asymmetric. For realistic applications of an adaptive scheme, a supervisory block is therefore mandatory. Some cases show a degradation of accuracy compared to standard OWICS, which is due to the amount of information gained from the number of acoustic paths of the ADM. For example, a four path ADM can yield four different velocities, which is a minimum of information to improve performance.

1 GAUSSIAN QUADRATURE

A quadrature rule is also known as numerical integration. Quadrature is often used, if the evaluation of the integral cannot be expressed in terms of standard functions. A famous case would be e^{-x^2} which is part of a Gaussian distribution. Quadrature can also be used for the integration of discrete data, which is the case for ADM.

There are two different types of quadrature to distinguish from. That is, Newton-Cotes quadrature and Gaussian quadrature. For Newton-Cotes quadrature, the ab-scissas zi are fixed. Usually they are uniformly spaced. For Gaussian quadrature, additional degrees of freedom are introduced by choosing optimal abscissas zi, such that the quadrature rule will result in the highest possible precision.

A Gaussian quadrature rule is an approximation to the general integral $\int_a^b f(x)\omega(x)dx$ where f(x) is some function and w(x) is termed the weight function. For the above mentioned famous case, the weight function would be w(x) = e^{-x^2}. Of course, the weight function may be any function suited to a particular problem, provided it corresponds to a family of orthogonal polynomials. Hence, if f(x) is a polynomial of degree less or equal to 2N−1, where N is the number of points used for the quadrature rule, Gaussian quadrature will perform exact integration due to the additional degrees of freedom.

1.1 Developing a Gaussian Quadrature Rule for ADM

A Gaussian quadrature rule is a weighted sum of function values, which need to be integrated. In order to determine the flow rate for an ADM, the area flow function need to be integrated. The following derivations are based on the OWICS method which is associated with the weight function w(x) = $(1- x^2)^k$. The family of orthogonal polynomials correlated to that weight function are called Jacobi polynomials. In terms of the quadrature rule, the weighted sum can be written as follows:

$$Q_{ADM} = \sum_{i=1}^{N} \omega_i F(z_i) \qquad \text{………..(1)}$$

where QADM is the volumetric flow rate calculated by the ADM in m^3s^{-1}, wi are the dimensionless weights according to the Gaussian quadrature rule, z_i are the abscissas according to the Gaussian quadrature rule in m and F(z) is the area flow function associated with m^2s^{-1}. For a conduit with circular cross section, the area flow function takes the form

$$F(z_i) = 2\sqrt{R^2 - z_i^2 \bar{v}_{ax}(z_i)} \qquad \text{………..(2)}$$

The radius of the conduit is denoted by R with units m and ¯vax(zi) = ¯vaxi is the mean axial velocity at the acoustic path position zi, which is the output of a N-path ADM (i = 1, 2, . . . ,N), in ms−1. Substituting Eq. (2) into Eq. (1) yields a Gaussian quadrature rule for ADM discharge calculation

$$Q_{ADM} = 2R\sum_{i=1}^{N} \omega_i \sqrt{R^2 - z_i^2}\, \bar{v}_{ax}(z_i) \qquad \text{………...(3)}$$

A complete derivation of Eq. (2) and Eq. (3) can be found in [8] and may be requested from one of the authors. Previous IGHEM contributions that also handle some of the content in this paper are [5] and [6].

1.2 Determination of the weights and Abscissas for a Gaussian Quadrature Rule

Basically there are two ways to determine the weights and abscissas for a Gaussian quadrature rule. One method uses the principle of undetermined coefficients, while the other, rather elegant method, uses the theory of orthogonal polynomials. As the method of undetermined coefficients will lead to a system of equations that can be used later on, both methods will be considered in this section.

1.2.1 Method of Undetermined Coefficients

The goal of a quadrature rule is to approximate the general integral $\int_a^b f(x)\omega(x)dx$ with a weighted sum of the form $\sum_{i=1}^{N} \omega_i'(x_i)$, i.e.,

$$\int_{-1}^{1} f(x)\omega(x)dx \approx \sum_{i=1}^{N} \omega_i' f(x_i) \qquad (4)$$

For simpler handling, the interval [a, b] is transformed into the symmetric interval [−1, 1]. The special benefit of a Gaussian quadrature rule is that if f(x) is a polynomial of degree less or equal to 2N−1, the weighted sum on the right side of Eq. (4) will be exactly equal to the integral. In order to determine the 2N unknowns, i.e., N weights ω_i' and N abscissas x_i, the function f(x) shall be a polynomial of a simple form. Thus, let

$$f(x) \equiv x^k, \qquad (5)$$

where k may be some integer greater or equal zero. Thus, Eq. (4) may be reformulated as follows

$$\int_{-1}^{1} x^k w(x)\,\mathrm{d}x = \sum_{i=1}^{N} w_i' x_i^k \qquad (6)$$

Eq. (6) is a system of k non-linear equations for all k > 1. Since Gaussian quadrature is of precision 2N −1, it follows that k must be between $0 \le k \le 2N-1$. This will yield 2N equations which can be solved for the 2N unknowns. After the solution has been found, the weights wi must be calculated according to the relation

$$w_i = \frac{w_i'}{w(x_i)} \qquad (7)$$

The down part of this approach is that it will always yield a system of non-linear equations for k>1, that will be ill-conditioned in general.

1.2.2 Orthogonal Polynomials

The inner product of two polynomials, ϕ_i and ϕ_i , is defined by the operator

$$\langle \phi_i, \phi_j \rangle = \int_a^b \phi_i(x)\phi_j(x)w(x)\,\mathrm{d}x \qquad (8)$$

It plays the central part for the calculation of the weights and abscissas with the theory of orthogonal polynomials.

A family of N + 1 polynomials ϕ_i for i = 0, 1, . . . ,N is said to be orthogonal if

$$\langle \phi_i, \phi_j \rangle = 0 \quad \forall i \neq j \qquad (9)$$

Then, the roots of _N are the abscissas xi, refer to § 6.6 of [1]. If the abscissas are determined, the weights can be calculated with

$$w_i = \frac{1}{w(x_i)} \int_a^b \ell_i(x) w(x)\,\mathrm{d}x \qquad (10)$$

The function `i(x) is the i-th Lagrange polynomial defined by

$$\ell_i(x) = \prod_{\substack{j=1 \\ j\neq i}}^{n} \frac{x - x_j}{x_i - x_j} \qquad (11)$$

Thus, the abscissas xi need to be known first in order to calculate the weights. Orthogonal polynomials satisfy a three term recurrence relation, that is,

$$\phi_{k+1}(x) = (x - \alpha_k)\phi_k(x) - \beta_k \phi_{k-1}(x) \qquad (12)$$

with ϕ_0 (x) = 1 and ϕ_{-1} (x) = 0. To find the abscissas xi, one need to find the roots of ϕ N. The Stieltjes algorithm can be used to calculate the orthogonal polynomial in an iterative manner up to the desired degree. The algorithm relates the recurrence coefficients k and k of Eq. (12) to the inner product, defined in Eq. (8) [2]. They can be calculated with the following equations

$$\alpha_k = \frac{\langle x\phi_k, \phi_k \rangle}{\langle \phi_k, \phi_k \rangle} \quad k \geq 0 \tag{13}$$

$$\beta_k = \frac{\langle \phi_k, \phi_k \rangle}{\langle \phi_{k-1}, \phi_{k-1} \rangle} \quad k > 0 \tag{14}$$

Where $\beta_0 = (\phi_0, \phi_0)$

A nice derivation of Eq. (12) may be found in [4] (although in a slightly different form). A proof for the above may be found in § 6.6 of [1]. A proof that the roots of ϕ_N are all real, simple and in the interval [a, b] may also be found in [1]. A list of common families of orthogonal polynomials and their corresponding weight function can be found in Table 6.4 of [1].

1.3 Example

The following example will show the advantages of a Gaussian quadrature rule applied to an ADM problem. The domain is a circular pipe with a diameter of D = 1 m. The ADM installation will consist of four paths, i.e., N = 4. In order to demonstrate the ability of a Gaussian quadrature rule to integrate without error, the disturbance function applied to the velocity profile must be a polynomial of degree less or equal to 2N −1. Figure 1 shows three OWICS based velocity profiles with different degrees of disturbance. Note that these disturbances may not be realistic but the example shall show that the mathematics hold, even for strong amplitudes. The exact discharge is calculated according to

$$Q_{ref} = \int_A \mathbf{u} \, \mathrm{d}\mathbf{A} \tag{15}$$

Table 1 Summary of calculated discharges according to Eq. (3) and Eq. (15)

Fig. 1	$w(x)$	$f(x)$	Q_{ref} [$\mathrm{m^3\,s^{-1}}$]	Q_{ADM} [$\mathrm{m^3\,s^{-1}}$]	ϵ [%]
(a)	$(1-x^2)^\kappa$	Poly. of deg. 0	2.8560	2.8560	6.0348×10^{-8}
(b)	$(1-x^2)^\kappa$	Poly. of deg. 4	2.6533	2.6533	6.4658×10^{-8}
(c)	$(1-x^2)^\kappa$	Poly. of deg. 8	3.0746	3.0525	-0.7184

The discharge according to the ADM is calculated with Eq. (3) for a four path OWICS system. The error between the two discharges is expressed with the following relation

$$\epsilon = \frac{Q_{ADM} - Q_{ref}}{Q_{ref}} \, 100\,\% \tag{16}$$

The results are summarized in Table 1. Note that the numerical integration with Eq. (3) introduces a large error for the case in Figure 1(c). Since N = 4, Gaussian quadrature is only exact up to polynomials with degree less or equal to seven. The polynomial disturbance in that case is of degree eight, hence, if a five path ADM (N = 5) would be used, exact integration for that case would be possible with a resulting error of $\epsilon = 8.1991 \times 10^{-8}$ %. The remaining small error is due to finite precision of the numerical calculations.

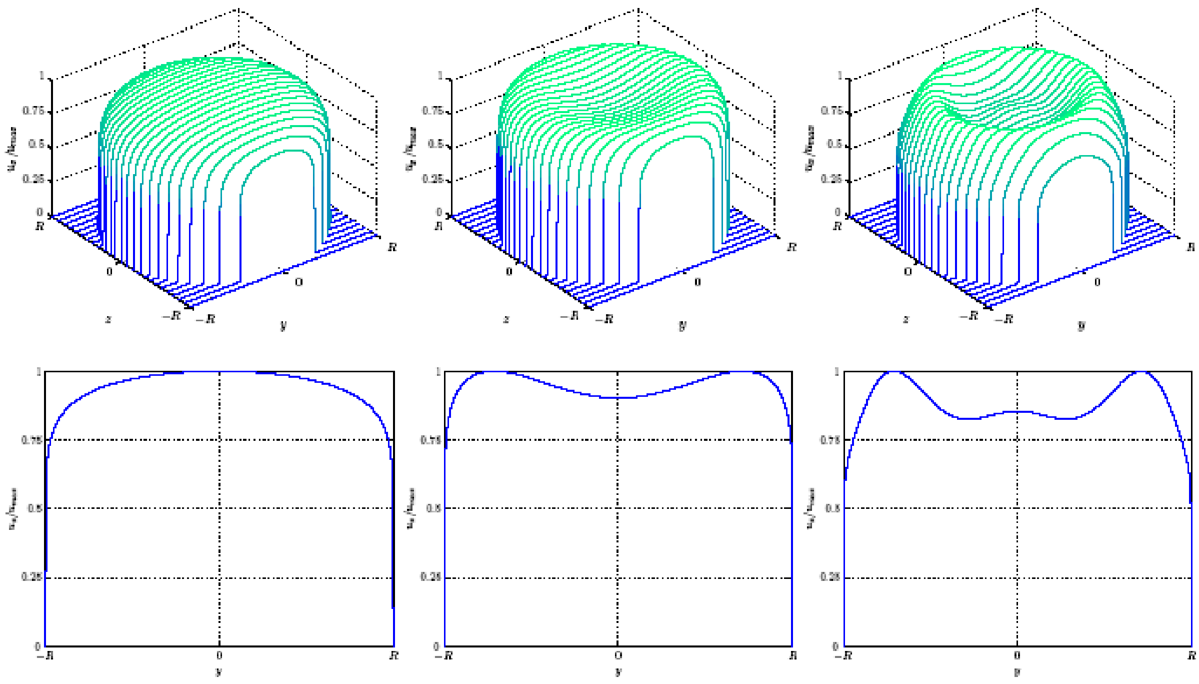

Figure 1 OWICS velocity profiles with different polynomial disturbances. The disturbance in column (a) is unity. In column (b), the disturbance function is a polynomial of degree 4 < 2N − 1. The disturbance in column (c) is a polynomial of degree 8 > 2N −1. The 2D plots illustrate the velocity profile at z = 0.

2 ADAPTIVE OWICS SCHEME FOR DISCHARGE CALCULATION

The fundamental part of the OWICS method, which also links the weight function of the Gaussian quadrature rule to the OWICS method, is the model of the velocity profile. It can be written in polar coordinates as follows

$$v_{owi} = v_{max,o}\left[1-\left(\frac{r}{R}\right)^2\right]^{\zeta} \tag{17}$$

Standard OWICS uses a fixed exponent κ = 0.6, which is part of the weight function. This exponent defines, together with the number of acoustic paths N, the weights and abscissas that are needed to apply Eq. (3). Tabulated values for wi and |zi| according to κ can be found in Table 2.4 of [6][1]. The exponent κ is directly linked with the exponent ζ of the velocity profile, i.e.,

$$\kappa \equiv \zeta + \frac{1}{2} \tag{18}$$

for a circular conduit. Since ζ describes the curvature of the velocity profile, it follows that if k is held constant, ζ must be constant too. This is a heavy restriction, as one velocity profile might be well suited for a particular flow situation, it may not be best suited for another flow situation. The idea of an adaptive scheme is to calculate ζ according to the measurement data provided by the ADM installation. This involves least squares regression with a fit model of the form of Eq. (17).

2.1 Linear Least Squares

As mentioned above, the fit model is of the same form as the velocity profile. Thus let

$$\hat{v}(q) = \alpha(1-q^2)^{\zeta}, \tag{19}$$

be the model velocity profile, where _ (defines the maximum velocity of the distribution) and ζ (defines curvature of velocity profile) are the fit parameters to be determined. The variable q is the dimensionless path position, which is equivalent to z/R. Such a fit model is of non-linear nature but it can be linearized by taking the natural logarithm on both sides of Eq. (19). Using least squares theory, the following linear system of equations need to be solved, in order to determine the fit parameters, where x = $[\ln(\alpha)\ \zeta\ ]^T$.

$$\begin{bmatrix} N & \sum_{i=1}^{N} \ln(1-q_i^2) \\ \sum_{i=1}^{N} \ln(1-q_i^2) & \sum_{i=1}^{N} \ln^2(1-q_i^2) \end{bmatrix} \mathbf{x} = \begin{bmatrix} \sum_{i=1}^{N} \ln(\bar{v}_{ax_i}) \\ \sum_{i=1}^{N} \ln(\bar{v}_{ax_i}) \ln(1-q_i^2) \end{bmatrix} \tag{20}$$

The value pairs ($\bar{v}_{axi}$, q_i) are the mean axial velocity, $\bar{v}_{axi}$, of the ADM at the dimensionless path position, $q_i = z_i/R$, respectively.

2.2 Adaptive Schemes

In this section, two adaptive schemes based on the OWICS idea are suggested to optimize discharge calculation in circular conduits.

2.2.1 Scheme Based on Weight Correction

This scheme uses the idea of corrected weights presented by Voser in [7]. However, the procedure to calculate the corrected weights will differ from [7]. Since the path positions stay unchanged, the abscissas zi = Rxi are known. Therefore, Eq. (6) can be used to solve for the corrected weights according to N paths. Since the abscissas[2] x_i are known, the originally non-linear system of equations turns into a linear system

$$Aw' = b \tag{21}$$

which can be solved for w' with common algorithms, such as Gaussian Elimination. The adaptive information, from the solution of the least squares problem, is contained in the weight function $w(x) = (1-x^2)^k$ of Eq. (6). That is, ζ only is needed for the weight correction scheme. The integral term on the left side of Eq. (6) can be expressed in an algebraic form with the Gamma function Γ

$$G_k(\varphi) = \int_{-1}^{1} x^k (1-x^2)^{\varphi}\, \mathrm{d}x = \cos^2\left(\frac{\pi}{2}k\right) \frac{\varphi\Gamma(\varphi)\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\varphi+\frac{k+3}{2}\right)} \tag{22}$$

Tabulated values of the Gamma function can be found in Appendix A. With the help of this relation, the linear system in Eq. (21) can be conveniently set up by

$$a_{ji} = x_i^{j-1} \tag{23}$$

$$b_j = G_{j-1}(\zeta + 1/2), \tag{24}$$

where a_{ji} are the elements of A, b_j are the elements of b and i, j = 1, 2, . . .N. Once the solution to the linear system, w', has been found, Eq. (7) must be applied element wise to get the actual weights w. The optimized discharge can then be calculated with Eq. (3) with the use of the corrected weights.

2.2.2 Scheme Based on Re-calculation of Weights and Abscissas

This scheme is not based on weight correction, but the re-calculation of both, the weights and abscissas. The re-calculation is due because the fitted value k = ζ + 1/2 might differ from standard OWICS, where k = 0.6. Thus, the theory in § 1.2.2 need to be applied. A very effective and stable numerical algorithm to calculate Gaussian quadrature rules is called the Golub-Welsch algorithm [3]. A Matlab code of the algorithm can be found in Appendix B. Once the weights and abscissas are re-calculated, the mean velocities $\bar{v}_{axi}$ from the ADM must be re-calculated as well. Since the abscissas xi will no longer be the same as before, the path positions will also differ and hence, different mean axial

[1] Note that some literature uses d_i instead of $|z_i|$.

velocities will result. To get over this problem, the velocity fit model of Eq. (19) with the calculated fit parameters can be used. The new mean velocities can then be calculated according to

$$\bar{v}_{ax_i} = \hat{v}(x_i) \tag{25}$$

Once all the re-calculations are done, the optimized discharge can be calculated with Eq. (3) and the newly determined values for wi, zi = Rxi and $\bar{v}_{axi}$

2.2.3 Procedure Summary

The following is a summary of the presented schemes above

Weight correction	**New weights and abscissas**
1. Perform least squares regression by solving Eq. (20).	1. Perform least squares regression by solving Eq. (20).
Set up Eq. (21) by using Eq. (23) and Eq. (24) with the previously determined fit paramater ζ from least squares regression. Apply Eq. (7) thereafter.	Determine the weights and abscissas according to § 1.2.2 with the previously determined fit parameter ζ from least squares regression, e.g., by using the Golub-Welsch algorithm. Depending on the algorithm used, the actual weights might be calculated with Eq. (7).
Determine the discharge with Eq. (3), using the corrected weights wi and the unchanged abscissas zi = Rxi as well as $\bar{v}_{axi}$	Re-calculate the mean axial velocities by using the fit model of Eq. (19) with the previously determined fit parameters α and ζ where $\bar{v}_{axi} = \hat{v}(x_i)$
	Determine the discharge with Eq. (3), using the all new values wi, zi = Rxi and $\bar{v}_{axi}$

2.3 Examples

2.3.1 Review of Example 1.3(c)

Refer back to the example (c) in Figure 1 on page 5. In this example, the adaptive schemes shall be applied and contrasted to the standard OWICS method as it was used in that previous example.

To get started, the linear system in Eq. (20) will be solved first in order to get the fit parameters α and ζ from the data provided by the ADM installation. By doing so, linear regression suggests ζ = 0.9813 ms^{-1} and ζ = 0.0218. This yields k = 0.5218, which is different from standard OWICS, but closer to the Gauss-Jacobi method[3]. Continuing from § 2.2.3, the resulting adaptive discharges can be calculated and are summarized in Table 2.

Note that the linear regression suggests k = 0.5218, which is less than that of standard OWICS. Using this information, gathered from this particular flow situation, the adaptive schemes perform more accurate compared to standard OWICS. Also, it is interesting that both adaptive schemes perfom equally.

[2]The difference between the ascissas x_i and z_i is that z_i has the dimension [m] associated with it and x_i is dimensionless. Thus, the abscissas x_i are always in the interval $[-1, 1]$.

Table 2 The table shows a summary of calculated discharges of Figure 1(c) in § 1.3. The discharges in rows two and three are calculated with the adaptive schemes discussed in § 2.2.1 and § 2.2.2, respectively. Note that they suggest k = 0.5218, which is less than standard OWICS, to yield a more accurate result compared to the standard method. Also note that both adaptive methods perform equal.

	Q_{ref} $[\mathrm{m}^3\,\mathrm{s}^{-1}]$	Q_{ADM} $[\mathrm{m}^3\,\mathrm{s}^{-1}]$	ϵ [%]	κ [−]
Standard OWICS	3.0746	3.0525	−0.7184	0.6000
Weight Correction		3.0573	−0.5628	0.5218
New Weights and Abscissas		3.0573	−0.5628	0.5218

2.3.2 Discharge Calculation Applied to CFD Simulation

The previous examples all featured deterministic velocity profiles. That is, they are described by Eq. (17) and a polynomial disturbance f(x). Thus, they are rotationally symmetric and share the same body structure of Eq. (17). In general, there is no such ideal and symmetric velocity profile, hence, ADM discharge calculation is applied to a CFD simulation in this last example, to examine the more general case. The domain of cal-

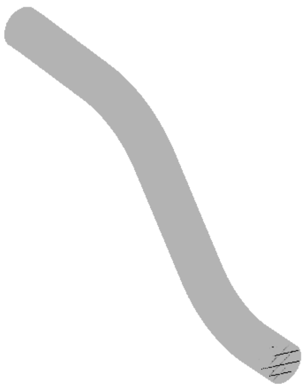

Figure 2 Domain of calculation for the CFD based example. The discharge is calculated at the indicated position of the acoustic paths of the ADM.

culation for this example is illustrated in Figure 2. It basically is a pipe flow with two bends in the same plane. The acoustics paths of the ADM are also shown. The two bends induce disturbances into the velocity profile, which is shown in Figure 3. The results are summarized in Table 3.

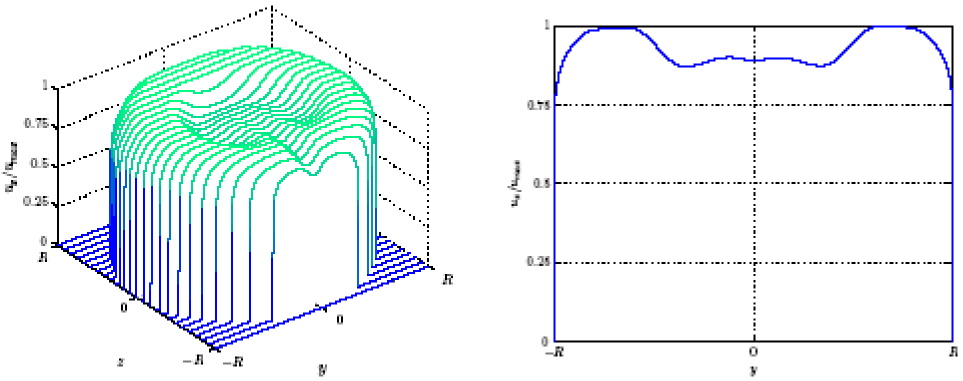

Figure 3 Velocity profile calculated with a CFD code at the intersection of the acoustic paths, shown in Figure 2.

[3]The GAUSS-JACOBI method is another integration method for discharge determination. It assumes a uniform velocity profile, thus, $\zeta = 0$ in Eq. (17). From there it follows that $\kappa = 1/2$.

Table 3 Summarized results of the discharge calculations based on a CFD simulation

	Q_{ref} [$\mathrm{m^3\,s^{-1}}$]	Q_{ADM} [$\mathrm{m^3\,s^{-1}}$]	ϵ [%]	κ [−]
Standard OWICS	941.2660	938.5185	−0.2919	0.6000
Weight Correction		939.9369	−0.1412	0.5237
New Weights and Abscissas		939.9369	−0.1412	0.5237

3 CONCLUSION

The previous examples have shown that a fixed k is not necessarily best suited for all flow situations. However, the standard OWICS method is optimized for use with a constant k = 0.6 (applies to pipe flows only), which yields approximate results for the discharge in the system. With the use of an adaptive scheme, as discussed in the previous sections, the accuracy of the discharge calculation can be improved further. The underlying velocity model of both, standard OWICS and the adaptive schemes, is given by Eq. (17). Since this velocity distribution is not a function of time, nor of the coordinate of main flow direction, it describes steady and fully developed flow, respectively. Hence, if an adaptive scheme is used with an ADM installation, where steady and fully developed flow along the pipe axis is given, the calculated discharge with the adaptive scheme will yield more accurate or equal results compared to standard OWICS. This may be the case for measurements in a straight pipe with a long enough inlet zone.

Due to limited space in practice, it is more common to install the transducers somewhere in the system, where the velocity profile is disturbed by bends in the main flow direction or other components such as fences or valves. For a standard four path installation (one cross plane) or an eight path installation (two cross planes intersecting at an angle to eliminate cross flow phenomena), a total of four data points are provided by the ADM for each cross plane. This velocity information is used in the adaptive schemes to calculate k from a two degree of freedom fit model. Since a least squares regression must be over-determined, three data points are required for a 2DOF fit model. Since a four path installation provides four data points, it is only one extra data pair available. This can be a limitation for heavily disturbed velocity profiles, meaning that the calculated _ from an adaptive scheme with four acoustic paths may yield worse results in terms of discharge as standard OWICS. Investigation showed that the adaptive scheme works more reliable on disturbances with somewhat symmetric character. For example, refer to Figure 3 on page 10. The velocity profile shown has a comparable symmetry about the center axis if contrasted to Figure 1(c), which is perfectly symmetric. Heavily disturbed velocity profiles with asymmetric characteristics may not yield an improved result compared to standard OWICS. This is due to the minimalistic velocity information used for least squares regression. Thus, further research is directed toward the improvement of performance of the fit model given in Eq. (19) and a supervision algorithm which controls the adaptive process.

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A. Gamma Function

The Gamma function is defined by

$$\Gamma(\varphi) = \int_0^\infty x^{\varphi-1} e^{-x}\,\mathrm{d}x \tag{26}$$

The following relations may be used to compute values for $\varphi < 1$ ($\varphi \neq 0,-1,-2,\ldots$) and $\varphi > 2$

$$\Gamma(\varphi) = \frac{\Gamma(\varphi+1)}{\varphi}, \quad \Gamma(\varphi) = (\varphi-1)\Gamma(\varphi-1)$$

φ	$\Gamma(\varphi)$	φ	$\Gamma(\varphi)$	φ	$\Gamma(\varphi)$	φ	$\Gamma(\varphi)$
1.00	1.00000	1.25	0.90640	1.50	0.88623	1.75	0.91906
1.01	0.99433	1.26	0.90440	1.51	0.88659	1.76	0.92137
1.02	0.98884	1.27	0.90250	1.52	0.88704	1.77	0.92376
1.03	0.98355	1.28	0.90072	1.53	0.88757	1.78	0.92623
1.04	0.97844	1.29	0.89904	1.54	0.88818	1.79	0.92877
1.05	0.97350	1.30	0.89747	1.55	0.88887	1.80	0.93138
1.06	0.96874	1.31	0.89600	1.56	0.88964	1.81	0.93408
1.07	0.96415	1.32	0.89464	1.57	0.89049	1.82	0.93685
1.08	0.95973	1.33	0.89338	1.58	0.89142	1.83	0.93969
1.09	0.95546	1.34	0.89222	1.59	0.89243	1.84	0.94261
1.10	0.95135	1.35	0.89115	1.60	0.89352	1.85	0.94561
1.11	0.94740	1.36	0.89018	1.61	0.89468	1.86	0.94869
1.12	0.94359	1.37	0.88931	1.62	0.89592	1.87	0.95184
1.13	0.93993	1.38	0.88854	1.63	0.89724	1.88	0.95507
1.14	0.93642	1.39	0.88785	1.64	0.89864	1.89	0.95838
1.15	0.93304	1.40	0.88726	1.65	0.90012	1.90	0.96177
1.16	0.92980	1.41	0.88676	1.66	0.90167	1.91	0.96523
1.17	0.92670	1.42	0.88636	1.67	0.90330	1.92	0.96877
1.18	0.92373	1.43	0.88604	1.68	0.90500	1.93	0.97240
1.19	0.92089	1.44	0.88581	1.69	0.90678	1.94	0.97610
1.20	0.91817	1.45	0.88566	1.70	0.90864	1.95	0.97988
1.21	0.91558	1.46	0.88560	1.71	0.91057	1.96	0.98374
1.22	0.91311	1.47	0.88563	1.72	0.91258	1.97	0.98768
1.23	0.91075	1.48	0.88575	1.73	0.91467	1.98	0.99171
1.24	0.90852	1.49	0.88595	1.74	0.91683	1.99	0.99581
1.25	0.90640	1.50	0.88623	1.75	0.91906	2.00	1.00000

B. The Golub-Welsch Algorithm

The following algorithm is called the Golub-Welsch algorithm according to [3]. The syntax of the code is Matlab R2009a. The arguments are N, k and μ_0, where $\mu_0 = G_0(k)$. Note that the weights returned are actually $\omega_i^{'}$, therefore, Eq. (7) must be applied thereafter.

```
function [xi,wi] = golub_welsch(n,kappa,mu0)
%calculate nodes xi and weights wi of the n point Gaussian quadrature rule
%with the weight function w(x) = (1-x^2)^kappa, based on the Golub-Welsch
%algorithm.
  J = zeros(n);
  xi = zeros(n,1);
  wi = xi;
  %compute Jacobi matrix (beta(i) = b(i) = 0)
  L = @(i)2*(i+kappa);
  alpha = @(i)(4*i*(i+2*kappa)*(i+kappa)^2/...
               (L(i)*(L(i)+1)*(L(i-1)+1)*(L(i-1)+2)))^(1/2);
  for i = 1:n
    if (i == 1)
      J(i,i+1) = alpha(i);
    elseif (i == n)
      J(i,i-1) = J(i-1,i);
    else
      J(i,i-1) = J(i-1,i);
      J(i,i+1) = alpha(i);
    end
  end
  [v,d] = eig(J); %solve eigenvalue problem
  for i = 1:n
    xi(i) = d(i,i);
    wi(i) = mu0*v(1,i)^2; %eigenvector must be normalized
  end
```