A METHOD FOR EQUATION DEDUCTION FOR EVALUATION OF RELATIVE MEASUREMENT UNCERTAINTY OF THE COMBINED AND UNCORRELATED VARIABLES

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ABSTRACT

Full measurement result is a set of quantity values being attributed to a measurand together with any other available relevant information. According to VIM [1] a measurement result is generally expressed as a single measured quantity value and a measurement uncertainty.

International comity for hydraulic efficiency measurement published a widely used international code IEC 60041:1991 [2]. It is dealing with hydraulic efficiency measurement which in Appendix A presents an evaluation of systematic uncertainties at measurement taken at steady state conditions. While the code states basic formulas for uncertainty calculations it does not state the basics and theory on how to deduct the uncertainty evaluation equations for setup different than those presented in the code.

In recent years new standards on measurement uncertainty vocabulary and assessment have been published from the Joint Committee for Guides in Metrology.

Hydraulic efficiency measurements are generally evaluated by a set of independent (uncorrelated) measured quantities, such as pressure or head measurements, discharge, temperature and generator power. Therefore, as the purpose of this paper is uncertainty assessment of hydraulic efficiency measurements, the paper focuses mainly on combined uncertainty theory for independent and uncorrelated measurements.

Based on GUM [3], effective and readable method to deduct equations for systematic measurement uncertainty evaluation is presented.

1. EVALUATION OF MEASUREMENT UNCERTAINTIES

Uncertainty of measurement is a parameter, associated with the result of a measurement, that characterizes the dispersion of the values that could reasonably be attributed to the measurand [4]. It is understood that the result of the measurement is the best estimate of the value of the measurand.

The uncertainty in the result of a measurement generally consists of several components which may be grouped into two categories according to the way in which their numerical value is estimated [5]:

A. those which are evaluated by statistical methods,
B. those which are evaluated by other means.

The components in category A are characterized and the estimated by the number of degrees of freedom. In general, where appropriate the covariance should also be given.

The components in category B should be characterized by terms, which may be considered approximations to the corresponding variances, the existence of which is assumed.

In laboratory and engineering practice, many physical quantities are, indirectly measured, by exploiting functional relations that connect them to other directly measured quantities [6]. With other words, indirectly measured values are calculated with combining several individually performed measurements.

It is obvious that uncertainties of the combined/indirectly measured values would in praxis not be evaluated by statistical methods (Type A), but are evaluated combining type B estimated individual measurement uncertainties.

1.1. Origin of the type B estimated uncertainties - Sensor uncertainty

To measure individual quantities, sensors and transducers (in continuation called sensors) are used. Each sensor that can be purchased on market comes with sensor specifications. The sensor specification contains data about sensor accuracy, repeatability, temperature influences, time drift etc.

Usually all datasheets state also a value that represent all uncertainty effects, however different terminology is used among the suppliers. The value representing all uncertainty effects may be called expected uncertainty, precision, sensor performance, or by other term. It is important that measurement engineer understands the meaning of the stated combined characteristic, which may refer to absolute or relative uncertainties stated for sensor range (full scale sometimes stated as *FS*).

For hydraulic efficiency measurements the biggest problem represents measurement of flow for which absolute uncertainty exceeds ±0,5% and can raise up to ±2% for some measuring method [2]. By evaluating combined measuring uncertainties we can conclude, that uncertainties smaller than ±0,2% contributes little to the final uncertainty, thus indicating that more precise uncertainty evaluation of individual measurements is unnecessary.

For each individual measurement individual sensor characteristics may be taken in account, however since todays sensor uncertainties are usually equal or better than ±0,2%FS, it hardly makes any sense to put tremendous effort in getting better uncertainty estimation.

2. THEORY ON THE COMBINED UNCERTAINTY EVALUATION

2.1. General theory on the combined uncertainty evaluation

In general we shall assume that measured quantities may be correlated. Assessment of combined uncertainty when quantities correlated is in general evaluated as according to GUM [3]:

$$u_c^{\,2} = \sum_{i=1}^{N} c_i^{\,2} u^2(x_i) + 2\sum_{i=1}^{N-1}\sum_{j=i+1}^{N} c_i \cdot c_j \cdot u(x_i)u(x_j)\cdot r(x_i, x_j) \qquad (1)$$

Where:

u_c …. Combined standard uncertainty

c_i ... **Sensitivity coefficient,** $c_i = \partial f / \partial x_i$

$u(x_i)$ ... Standard uncertainty of variable x_i

$r(x_i, x_j)$ ... Correlation factor between variables x_i and x_j

For a special case where all quantities are uncorrelated, meaning that correlation coefficients are equal to $r(x_i, x_j) = 0$, the equation simplifies to

$$u_c^{\,2} = \sum_{i=1}^{n} \left(c_i \cdot u(x_i)\right)^2 = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i} u(x_i) \right)^2 \qquad (2)$$

There is inconsistency between the usage of designation for uncertainty terms between GUM and IEC 60041 [2]. Since traditionally IEC designations are widely used in uncertainty evaluations in hydraulic efficiency measurements, IEC designations will be used in the text.

2.2. Combined uncertainty evaluation for uncorrelated uncertainty contributing variables

It has been explained that in hydraulic efficiency measurements mostly all measured quantities are uncorrelated or correlation factors are close to 0 and as such assumed as uncorrelated. In this case, equation (2) may be used for combined measurement uncertainty evaluation.

2.2.1. Absolute uncertainty e_x of the measured Variable X

Measurements are to be expressed as $X_o \pm \delta X$, where X_o is central measured value and δX is absolute uncertainty which has the same dimension as the central measured value [6]. According to IEC60041, the absolute uncertainty is marked as e_X.

Example - Pressure measurement result with its measurement uncertainty is put down as:

$$p = p_o \pm e_p = 41{,}5bar \pm 0{,}05bar \qquad (3)$$

2.2.2. Combined absolute measurement uncertainty e_F of the measured combined variable *F*

For the function *F* which is combined of *i* independent (uncorrelated) variables x_i, the evaluated combined absolute uncertainty can be expressed as:

$$e_F = \sqrt{\sum_{i=1}^{n} \left(\frac{\partial F}{\partial x_i} e_{x_i} \right)^2} \qquad (4)$$

where $F = f(x_1, x_2, x_3, ...)$.

2.2.3. Relative measurement uncertainty f_X of the measured variable X:

The quality of a measurement cannot be solely determined by its absolute uncertainty e_x. For example, an absolute uncertainty e_x = 1mm has different meaning when referring to a length X_o = 1 cm or to a length X_o = 10 m.

The quality of a measurement is better expressed by the relative uncertainty which is expressed as a fraction between the absolute uncertainty e_X and absolute value of measured value X:

$$f_X = \frac{e_x}{|X|} \, [/] \qquad (5)$$

The smaller the relative uncertainty is, the higher the quality of the measurement. By definition, the relative uncertainty is always a dimensionless quantity [6].

2.2.4. Combined relative measurement uncertainty for uncorrelated variables x_i

$$f_F = \sqrt{\sum_{i=1}^{n} f_i^{\,2}} \quad [/] \qquad (6)$$

where $F = f(x_1, x_2, x_3, ...)$ and f_i is evaluated uncertainty of the independent variable x_i.

2.3. Deduction of equations for the evaluation of combined measurement uncertainty for uncorrelated variables

2.3.1. Function F is a multiplication or division of independent variables

Let's assume a function F which is a product of two independent (uncorrelated) variables A and B:

$$F = A \cdot B \qquad (7)$$

According to equation (4), the combined measurement uncertainty is:

$$e_F = \sqrt{\left(\frac{\partial F}{\partial A} \cdot e_A\right)^2 + \left(\frac{\partial F}{\partial B} \cdot e_B\right)^2} \qquad (8)$$

After solving the partial derives, the equation for combined absolute uncertainty becomes:

$$e_F = \sqrt{(B \cdot e_A)^2 + (A \cdot e_B)^2} \qquad (9)$$

If we use equation (9) for absolute measurement uncertainty of the combined variable F and divide it by its absolute value, than we get the combined relative uncertainty as:

$$f_F = \frac{e_x}{|X|} = \frac{\sqrt{(B \cdot e_A)^2 + (A \cdot e_B)^2}}{|A \cdot B|} = \sqrt{\left(\frac{B}{A \cdot B} \cdot e_A\right)^2 + \left(\frac{A}{A \cdot B} \cdot e_B\right)^2} \qquad (10)$$

If individual absolute measurement uncertainties e_A and e_B are calculated from the expected relative measurement uncertainty of the used equipment based on equation (5), than the combined relative uncertainty can be written as:

$$f_F = \sqrt{\left(\frac{1}{A} \cdot f_A \cdot A\right)^2 + \left(\frac{1}{B} \cdot f_B \cdot B\right)^2} = \sqrt{f_A^2 + f_B^2} \quad (11)$$

This mathematical example shows that combined relative uncertainty of two products of two independent variables can be calculated as the square roots from the sum of squares of individual variables.

When function F, is combined from product of independent variables x_i as:

$$F = x_1 \cdot x_2 \cdot ..... \cdot x_i \quad (12)$$

than combined uncertainty f_F can be in general form written as:

$$f_F = \sqrt{\sum_{i=1}^{n} f_i^2} \quad (13)$$

Deduction for fractional functions (not presented in this paper) shows that principle from equation (13) can exactly be used for evaluation of combined relative uncertainty of the fractional functions.

2.3.2. Function F is an addition or subtraction of independent variables

Let's assume a function H which is a sum several independent (uncorrelated) variables hi:

$$H = h_1 + h_2 + ... + h_i \quad (14)$$

According to equation (4), the combined measurement uncertainty is:

$$e_H = \sqrt{\left(\frac{\partial H}{\partial h_1} \cdot e_{h_1}\right)^2 + \left(\frac{\partial H}{\partial h_2} \cdot e_{h_2}\right)^2 + ... + \left(\frac{\partial H}{\partial h_i} \cdot e_{hi}\right)^2} \quad (15)$$

If we solve partial derives and use equation (9) in a way to deduct relative measurement uncertainty than we receive

$$f_H = \frac{e_H}{H} = \frac{\sqrt{(h_1 \cdot f_{h1})^2 + (h_2 \cdot f_{h2})^2 + ... + (h_i \cdot f_{hi})^2}}{h_1 + h_2 + ... + h_i} \quad (16)$$

With further deduction the equation becomes:

$$f_H = \sqrt{\left(\frac{h_1}{H} \cdot f_{h_1}\right)^2 + \left(\frac{h_2}{H} \cdot f_{h_2}\right)^2 + ... + \left(\frac{h_i}{H} \cdot f_{h_i}\right)^2} \quad (17)$$

For a function H which is a sum several independent (uncorrelated) variables h_i

$$H = \sum h_i \quad (18)$$

the general formula for evaluation of combined (weighted) relative uncertainty is:

$$f_H = \sqrt{\sum \left(\frac{h_i}{H} \cdot f_{h_i} \right)^2} \tag{19}$$

2.3.3. Function F is a combined from independent variables where one of the variables is powered

Let's assume a function F which is a product several independent (uncorrelated) variables hi:

$$F = A^n \cdot B \tag{20}$$

According to equation (4), the individual measurement uncertainty of the term A^n is:

$$e_{A^n} = \frac{\partial F}{\partial A} \cdot e_A \tag{21}$$

After solving the partial derives, the equation for combined absolute uncertainty becomes:

$$e_{A^n} = B \cdot n \cdot A^{n-1} \cdot e_A \tag{22}$$

If we use equation (9) for absolute measurement uncertainty of the combined variable F and divide it by its absolute value, than:

$$f_{A^n} = \frac{e_{A^n}}{\left| A^n \right|} = \frac{B \cdot n \cdot A^{n-1} \cdot e_A}{A^n} = \frac{B}{A^n \cdot B} \cdot \frac{A^n}{A} \cdot n \cdot e_A = \frac{1}{A} \cdot n \cdot e_A \tag{23}$$

When absolute uncertainty e_A is calculated from the expected relative measurement uncertainty of the equipment based on equation (5), than the relative uncertainty can be written as:

$$f_{A^n} = \frac{1}{A} \cdot n \cdot f_A \cdot A \tag{24}$$

After we reorder equation (24) we receive general formula for evaluation of the relative uncertainty of powered term:

$$f_{A^n} = n \cdot f_A \tag{25}$$

2.3.4. Function F is a multiplication or constant and measured variable

$$F = k \cdot A \tag{26}$$

$$f_F = \sqrt{\sum_{i=1}^{n} \left(\frac{\partial F}{\partial x_i} \right)^2} = \sqrt{\left(\frac{\partial F}{\partial A} \right)^2} = k \cdot f_A \tag{27}$$

$$e_F = \sqrt{\sum_{i=1}^{n} \left(\frac{\partial F}{\partial x_i} x_i \right)^2} = \sqrt{\left(\frac{\partial F}{\partial A} A \right)^2} = k \cdot f_A \cdot A = k \cdot e_A \tag{28}$$

When measured variable is multiplied by a constant, a relative or an absolute uncertainty of the function is a multiplication of variable uncertainty.

3. METHODOLOGY FOR SIMPLIFIED DEDUCTION OF EQUATIONS FOR MEASUREMENT UNCERTAINTY EVALUATION

Let's consider some example equation of efficiency measurement by ultrasonic measurement. We assume that all variables are independent:

$$\eta_t = \frac{Pt}{\rho \cdot g \cdot H_n \cdot Q_t} = \frac{P_{gen} / \eta_{gen}}{\rho \cdot g \cdot (H_1 - H_2) \cdot Q} \qquad (29)$$

To evaluate measurement uncertainty GUM proposes to calculate sensitivity coefficients c_i for each measured quantities and calculate combined uncertainty according to eq (2). However for complex equations this method can be as first mathematically demanding and presentation of partial results difficult (partial results would in equation (29) refer to uncertainty of P_t and H_n. Note that calculation sensitivity coefficient only represents a sensitivity or influence of particular measured quantity on the final result.

We propose that a process of equation deduction is better to be done in sequence, where we separately evaluate and combine uncertainties for each part of measurement and calculation.

To evaluate measurement uncertainty we propose to use a **methodology of "Order of mathematical operations**" know in mathematics as arithmetic precedence. So, in a same way as we start calculating above example as (H_1-H_2) in the first step, we would also start determining uncertainty of measurement of net head (H_1-H_2). In second step we would calculate uncertainty of flow measurement and so on. The idea is, that if a part of equation represents some physical value (like H_n in eq. (29) does), we would like to present it in our assessment of measurement uncertainty.

Once we know where to start, we continue by deduction of the equation parts. The deduction of equations for uncertainty evaluation can be performed as follows, depending of the pattern of analyzed portion of the equation:

a.) X = A1 + A2 + … → for uncertainty evaluation use rule (19)

b.) X = A1 * A2 *… → for uncertainty evaluation use rule (13)

c.) $X = A^n$ → for uncertainty evaluation use rule (25)

d.) X = k*A → for uncertainty evaluation use rule (28)

3.1. Uncertainty assessment example

Consider measurement of turbine efficiency for Kaplan unit:

- Flow is determined by Winter Kennedy method; spiral chase pressure difference at Winter-Kennedy pressure taps Δp_{WK} is measured by 0÷0,5bar differential meter, with relative uncertainty f=0,2% FS.
- Net head is measured as a level difference between H1 and H2 taking into account the dynamic heads at measured points:
  - H1 is measured by pressure transducer (0÷1bar, f=±0,15% FS) installed behind trash-racks; measuring crossection A2 estimated at f =±10%
  - H2 is measured by Plant measurement with checked uncertainty of e=±5cm (taking into account waves at tail-water level); measuring crossection A2 estimated at f =±5%
- Generator power is measured by means of electrical power measurement using current and voltage transformers, both class 0,5%; power meter uncertainty is f=±0,2% on measured value.
- Turbine power is measured as generator power using generator efficiency characteristic (estimated uncertainty at f=±0,2%); Generator bearing losses are included in the generator efficiency characteristic. Turbine guide bearing losses are estimated by calculation with expected uncertainty f=±25%.
- Uncertainties in water density as well as gravity factor are neglected.

Uncertainty assessment for turbine efficacy is first divided into separate analysis for uncertainties of flow, head and turbine power.

According to IEC60041:1991, the flow measurement uncertainty by Winter-Kennedy (WK) is estimated to be f=~±1,5%; Therefore we can assume that uncertainty of determining of WK constants (k and n) is estimated to be within f = ±1,5%. Flow uncertainty is than evaluated using eq. (25), and simplification for WK uncertainty estimate:

$$Q = k \cdot \Delta p_{WK}{}^{n} \tag{30}$$

$$f_Q = \sqrt{n \cdot f_{dpWK}{}^2 + f_{WKconst}{}^2} \tag{31}$$

Whenever values are measured by a transducer that have a relative uncertainty characteristic stated compared to full scale range (FS), first absolute measurement transducer uncertainty is to be determined and then compared to the measured value. By following this procedure, relative uncertainty of the measurement taken is calculated. In the Table 1, the measurement uncertainties of differential pressure dp_{WK} and intake pressure behind trashracks p_{HWLtr} are evaluated.

In H_n uncertainty evaluation, first Static Head and then Dynamic heads are evaluated.
Net head is determined as:

$$H_n = H_{stat} + H_{dyn} = (H_{stat1} - H_{stat2}) + (H_{dyn1} - H_{dyn2}) \tag{32}$$

Dynamic head is calculated at measured points is calculated using discharge and known area, and so:

$$H_n = H_{stat} + H_{dyn} = \left(HWL_{tr} - TWL\right) + \left(Q_t^2 / 2g \cdot \left(1/A_1^2 + 1/A_2^2\right)\right) \tag{33}$$

Net head uncertainty evaluation in three steps:

$$f_{Hstat} = \sqrt{\tfrac{HWLtr}{Hstat}\, f_{HWLtr}{}^2 + \tfrac{TWL}{Hstat}\, f_{TWL}{}^2} \tag{34}$$

$$f_{Hdyn} = \sqrt{2 \cdot f_Q{}^2 + (\tfrac{Hdyn1}{Hdyn} \cdot 2 \cdot f_{A1}{}^2 + \tfrac{Hdyn2}{Hdyn} \cdot 2 \cdot f_{A2}{}^2)} \qquad (35)$$

$$f_{H_n} = \sqrt{\tfrac{Hstat}{Hn}\, f_{Hstat}{}^2 + \tfrac{Hdyn}{Hn}\, f_{Hdyn}{}^2} \qquad (36)$$

Developed example of turbine efficiency procedure is presented in Table 1.

$$f_{\eta_t} = \sqrt{f_Q{}^2 + f_{H_n}{}^2 + f_{P_t}{}^2} \qquad (37)$$

Table 1 – Measurement uncertainty evaluation example

Measurement point				1	2
		designation		uncertainty (+-)	
Flow:					
Differential pressure					
relative uncertainty	pressure gauge relative uncertainty	f_dp	%	0,2	
/	dp gauge meter range	r_dp	mbar	250	
absolute uncertatinty:	absolute uncertainty	e_dp	mbar	0,5	
/	measured dpWK	dpWK	mbar	110	
relative uncertainty	**uncertainty of dpWK**	**f_dpWK**	**%**	**0,45**	
Flow					
relative uncertainty	Winter Kennedy constants estimation uncertatinty	f_Wkconst	%	1,50	
/	Winter Kennedy exponent factor	n	/	0,500	
relative uncertainty	**Flow estimation uncertainty**	**f_Q**	**%**	**1,53**	
Net Head	measured values:	*Hn*	*m*	*15,6*	
Intake level					
/	pressure gauge range	*r_p_HWL*	*bar*	*1,6*	
relative uncertainty	pressure gauge uncertainty	*f_p_HWL*	*%*	*0,15*	
absolute uncertatinty:	high water level after trashracks	e_HWLtr	m	0,024	
Static Head		*Hstat*	*m*	*15,3*	
relative uncertainty	high water level after trashracks (compar. to Hstat)	f_HWL_tr	%	0,16	
absolute uncertatinty:	tail water level	e_TWL	m	0,05	
relative uncertainty	tail water level (comparing to Hstat)	f_TWL	%	0,33	
relative uncertainty	**static head (compared to Hn)**	**f_Hstat(Hn)**	**%**	**0,36**	
Dynamic Head	Measured dynamic head	*Hdyn*	*m*	0,20	
	Measured dynamic head at point 1	*Hdyn1*	*m*	0,32	
	Measured dynamic head at point 2	*Hdyn2*	*m*	0,12	
relative uncertatinty:	A1 crossection estimation	f_A1	**%**	10,0	
relative uncertatinty:	A2 crossection estimation	f_A1	**%**	5,0	
relative uncertatinty:	flow measurement	f_Q	**%**	1,53	
relative uncertatinty:	**dynamic head (comparred to Hn)**	**f_Hdyn(Hn)**	**%**	**0,24**	
Net Head					
relative uncertainty	**Net Head**	**f_Hn**	**%**	**0,43**	
Turbine power measurement		*Pt*	*MW*	*25,00*	
Generator power					
relative uncertainty	Power meter uncertainty	f_P	%	0,20	
relative uncertainty	Voltage transformer class	f_CT	%	0,50	
relative uncertainty	Current transformer class	f_PT	%	0,50	
relative uncertainty	**Generator power**	**f_Pgen**	**%**	**0,73**	
relative uncertainty	generator efficiency uncertainty estimation	f_eta_gen	%	0,20	
Bearing losses		*TB_loss*	*kW*	15,00	
relative uncertainty	turbine bearing losses estimation	*f_TB_loss_est*	*%*	25,00	
relative uncertainty	**turbine bearing losses calculated for Pt**	***f_TB_loss***	*%*	**0,02**	
Turbine power					
relative uncertainty	**Turbine power**	**f_Pt**	**%**	**0,76**	
Hydraulic efficiency		eta_t	**%(rel)**	**93,40**	
relative uncertainty	turbine efficiency	f_eta_t	**%**	**1,77**	
absolute uncertainty	turbine efficiency	e_eta_t	**%(abs)**	**1,65**	

4. CONCLUSIONS

Methodology presented in this paper offer simple method for deduction of equations for evaluation of relative measurement uncertainty for variables combined by independent (uncorrelated) individual measurements. Methodology showed to be useful for simple as well as complex variable equations.

By performing measurement uncertainty evaluation by partial evaluation as per presented methodology, individual contributions of certain parts on the combined uncertainty can be evaluated and presented. This way of uncertainty analysis can highlight parts that have most influence on the measured result which is a basis for evaluation of possible and meaningful improvements for achieving lower measurement uncertainty.

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